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AN IMPROVED INDEX AND ESTIMATION METHOD FOR ASSESSING TAX PROGRESSIVITY

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Abstract

Amid the recent debates about federal tax policy fairness, we critically compare various measures of tax progressivity and the methodology used to estimate their value with empirical data. First, we propose criteria for properly measuring tax progressivity and apply them to these measures. Next, we propose criteria for evaluating the process of estimating these measures with data on the distribution of income earned and taxes paid. Last, we examine these various methods of measuring tax progressivity using an example dataset to reveal the differences in tax-progressivity values produced by these various progressivity measures. The analysis as a whole identifies a superior progressivity measure and estimation methodology that can be applied to a more comprehensive set of income and tax-burden distribution data to reveal a consistent and accurate measure of federal tax policy progressivity. This index is capable of producing testable claims on the degree of progressivity, where these test results can edify the normative federal tax policy debate.

***JEL* codes:** H2, H3

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An Improved Index and Estimation Method for Assessing Tax Progressivity

Michael D. Stroup and Keith Hubbard

I. Introduction

One of the most contentious tax policy issues during the 2012 presidential election season involved the federal income tax rate reductions defined in the Jobs and Growth Tax Relief Reconciliation Act of 2003. Signed into law by President George H. W. Bush, the act lowered the marginal income tax rate structure across all income levels but was set to expire on January 1, 2011. The candidates of both parties debated whether the act should be extended, modified, or left to expire and revert to the previous, higher marginal tax rate structure. Concerns over the US economy's faltering recovery after a deep recession pushed Congress to pass, and President Obama to sign, the Tax Relief, Unemployment Insurance Reauthorization and Job Creation Act, which extended many of the Bush tax cuts until then end of 2012. Then on January 2, 2013, President Obama signed a last-minute "fiscal cliff" tax bill produced by Congress that, among other things, extended the lower federal income tax rates for all but the 1 percent of US households with the highest incomes (as defined by the latest IRS data on the adjusted gross incomes of all taxpaying US households). The debate over federal income tax burden fairness continues today.

Many aspects of the debate have involved contradicting claims that were, ostensibly, empirically testable. Examples include whether the lower marginal income tax rates increased or decreased total income tax revenues via a supply-side effect, or whether the lower income tax rates encouraged more or less economic growth in the long run. However, the most volatile aspects of the debate have concerned whether upper-income American households were shouldering their fair share of the total federal tax burden. Indeed, the media have given widespread attention to the increasing gap between the highest- and lowest-income groups in American society, which suggests that federal tax policy fairness is an issue of particular interest to voters.

Whether a given income tax scheme is "fair" is a decidedly normative question. However, analysis with positive claims about competing federal income tax schemes can be developed to enlighten this normative debate. While previous attempts have been made to compare the individual ability of American taxpayers to pay their taxes with the actual tax burdens they face, they often fall short of their goal because the issue of assessing whether upper-income households are shouldering an appropriate level of the federal income tax burden is complicated. Americans face a complex maze of federal taxes, including income taxes, payroll taxes, corporate income taxes, estate taxes, excise taxes, and more. Further, it is difficult to measure each American's ability to pay these taxes. Should tax policy makers be concerned with the tax burden that each individual will bear over a lifetime? Should the stock of an individual's personal wealth be added to the flow of personal income when assessing the fairness of his or her tax burden? These are challenging questions. Yet, once a consensus on the proper set of overall income and tax-burden distribution data is reached, there remains the difficult task of properly comparing income and tax-burden distributions in a clear and testable manner.

In this light, economists have developed the concept of tax progressivity. A given tax rate scheme is effectively progressive if a person's average tax rate on income increases with the level of his or her income. This means that over time, a tax structure has become more progressive if the ratio of income used to pay federal taxes rises among higher-income earners or falls among lower-income earners. The *degree* to which a proposed tax scheme shifts a greater share of this tax burden onto the higher-income groups should be a quantifiable proposition that can meaningfully inform the normative debate on federal tax policy fairness, provided a proper dataset is chosen and an appropriate index is developed.

The goal of this paper is not to answer the normative question of whether America's federal tax scheme is fair. Nor is it to identify the proper dataset to employ in this endeavor. Rather, we set out to establish the best way to effectively measure tax progressivity, which is an important step toward enlightening the ongoing debate about federal tax policy fairness. The following is a proposition for determining the best methodology and the resulting index for measuring the degree of an income tax scheme's progressivity, as well as an improved method for estimating such indexes from empirical data comparing the income and tax-burden distributions across an entire tax base.

First, we propose a set of qualitative principles for critically evaluating any tax-progressivity methodology that can produce measures of progressivity, referred to hereafter as indexes. We apply these principles to two well-established indexes (Kakwani 1977; Suits 1977) and to a more recent index (Stroup 2005). We also examine the informal method of analyzing progressivity developed by Piketty and Saez (2007). We show that the Stroup index arises from a methodology that is superior to these others, making it a more accurate and more reliable index of overall tax progressivity.

Next, we propose a set of principles for evaluating any process to estimate tax progressivity using income and tax-burden distribution datasets. We apply these principles to well-established estimation procedures as well as to a revised method of estimation. We use publicly available IRS data to illustrate the properties of the different estimation methods and find that this revised process is superior for estimating any tax-progressivity index.

Finally, we use annual IRS data from the last quarter century to estimate and observe the behavior of the three income tax progressivity measures over time. We show how the Stroup index indicates that federal income tax progressivity has increased over time, whereas the other three methodologies imply that income tax progressivity has declined. We note that this disparity may not be driven by the choice of data used in their analysis, but may be the result of inherent design flaws in the three methodologies other than Stroup's. Further, we conclude that the Stroup index could accurately reflect the overall federal tax burden by using a more comprehensive income and tax-burden distribution dataset, such as that used by Piketty and Saez. We conclude that in this case, the Stroup index would provide a cardinal measure of overall tax progressivity that is unbiased, comprehensive, and reliable.

II. Why the Fairness Debate Needs a Concise Index for Progressivity

Scanning the recent editorial pages of major newspapers and popular political blogs reveals that the fairness debate over our federal tax structure continues to be a major concern. Those who wish to make permanent the lower marginal income tax rate scheme of the Bush tax cuts may cite how higher-income groups in the United States are bearing an increasingly greater share of the total federal income tax burden after the inception of this act. This view implies that the federal income tax system has become effectively more progressive, despite the lower marginal income tax rate structure that this act has levied on the tax base over the last decade. Conversely, those who desire the return to a higher marginal income tax rate structure may cite how the upper-income groups have been earning an ever-larger share of the total income earned in the economy over the last decade. They claim that the lower marginal tax rate structure has made the federal income tax structure less progressive, despite the greater share of total income tax revenue paid by those with the highest incomes.

This ongoing debate may be at a rhetorical impasse, with each side failing to comprehend the other's fundamental arguments and weigh them properly. There appear to be at least two reasons for this: (1) changes in tax-burden shares and changes in income shares are often considered independently when they should be considered simultaneously, and (2) even when these changes are considered simultaneously, the impact on the highest-income earners alone is often used to determine the degree of tax progressivity for an entire tax scheme without considering the impact on the entire tax base.

To illustrate the first reason for this rhetorical impasse, consider Carroll (2009), who argues that lower marginal tax rates would effectively increase progressivity of the federal income tax base. He notes that the economically unproductive activities of tax avoidance (legal) and tax aversion (illegal) among upper-income households become less profitable with lower tax rates. He estimates that the resulting increases in economic productivity would raise the share of the total federal income tax burden borne by the rich.

Indeed, IRS data from reveal that the share of the federal income tax burden borne by the top 1 percent of households, as measured by adjusted gross income (AGI), rose from 33.7 percent to 36.7 percent 2002–2009. Over the same period, the share of the tax burden borne by the lower 50 percent of US households fell from 3.5 percent to 2.3 percent (Logan 2011). However, this perspective fails to enlighten the fairness debate because looking at the tax-burden distribution alone ignores any concomitant changes in income distribution. If the income shares earned by those with the highest incomes have also increased, those taxpayers would need to pay a larger share of the tax burden to maintain the same level of tax progressivity.

Conversely, Gale and Orszag (2005) predict that lower marginal tax rates effectively decrease the progressivity of the federal income tax base because they would transfer income and wealth from poor and middle-class households to higher-income households. IRS data also support this view, revealing that the share of income earned by the top 1 percent of US households rose from 16.1 percent of total AGI to 16.9 percent over the same 2002–2009 period. Meanwhile, the share of total AGI earned by the lowest 50 percent of the population fell from 14.2 percent to 13.5 percent (Logan 2011). However, this perspective fails to enlighten the fairness debate because looking at the income share distribution alone ignores concomitant changes in the relative tax-burden shares borne by each income group. If the tax-burden share paid by the rich has also increased, the rich would need to earn a larger income share to maintain the same level of tax progressivity.

To illustrate the second reason for this rhetorical impasse, consider the much-cited article on federal tax policy fairness by Piketty and Saez. They aggregate a set of federal tax categories (federal income taxes and payroll taxes, corporate income taxes, and excise taxes) to represent the total federal tax burden. They then compare the pre- and post-tax income shares for each income segment of the population to calculate an average federal tax rate. They confirm that the federal tax system is progressive by showing how higher-income groups each bear a progressively bigger decline in their after-tax income shares.

However, Piketty and Saez do not propose any cardinal measure by which to quantify the *degree* of tax progressivity, making it difficult to interpret their results. To illustrate, they produce a pair of charts comparing the average federal tax rate facing the various income percentiles of the US population in 1960 and in 2004. They note that the average federal tax rate of the top 1 percent of income earners fell from the mid–70 percent range in 1960 to the mid–30 percent range in 2004. They state that federal tax policy became much less progressive over that time period (Piketty and Saez 2007, 11–12).

Yet this same pair of charts reveals that taxpayers in the 20th to 40th percentiles also enjoyed a decrease in their average federal tax rates, from about 13 percent in 1960 to about 9 percent in 2004, which would *increase* the degree of tax progressivity. Was the tax-burden decline among the few people in the 1 percent of income earners of sufficient magnitude to overshadow the tax-burden decline enjoyed by the many people in the 20th to 40th percentiles? Piketty and Saez claim that because the income earned by the top 1 percent of taxpayers represents a larger proportion of total national income, their influence on the degree of tax progressivity measure should predominate. Yet how do we *quantify* the net change in tax-scheme progressivity over time without using income information from the entire tax base?

This discussion makes it clear that an informal, nonparametric method like that used by Piketty and Saez produces few testable conclusions about tax-progressivity changes over time and therefore fails to enlighten the federal tax policy fairness debate. This debate needs an easily understood and widely trusted index of tax progressivity that consistently produces a cardinal value reflecting the relative degree of overall tax-scheme progressivity. In the next section, we carefully address this issue as a first step toward creating an objective tax-progressivity index and therefore a more edifying dialogue about federal tax policy fairness.

III. A Methodology for Properly Interpreting Tax Progressivity

Kakwani (1977), Suits (1977), and Stroup (2005) have all developed separate but conceptually related indexes that attempt to measure income tax progressivity across the entire tax base. We estimate the values of these three indexes using publicly available IRS data on cumulative federal income tax and AGI distributions in the United States (Logan 2011). These annual index values are reported in table 2, which appears in a later section where we examine these values in greater depth. This income and tax dataset is used mainly to illustrate and compare the behavioral characteristics of all three indexes, rather than to produce any definitive claims about federal tax-burden progressivity. Table 2 reveals that the annual values of these three indexes often diverge qualitatively over time. Which index most accurately reflects the true change in the degree of tax progressivity of a given tax scheme? The answer requires a closer look at how each index is designed.

As mentioned earlier, the traditional definition of tax progressivity is when a tax scheme produces an effective average tax rate that increases with income. This means that a progressive tax scheme causes individuals with greater incomes to pay a disproportionately higher share of their income in taxes. Also, the extreme ends of the tax-progressivity spectrum can be well defined. The lowest degree of tax progressivity possible without becoming a regressive tax scheme is a proportional income tax—sometimes called a "flat tax"—where all taxpayers pay the same percentage of their income in tax regardless of their income level. The highest degree of tax progressivity occurs when the single individual with the greatest income in the tax base bears the entire tax burden.

Comparing the degree of tax progressivity in competing tax schemes, or tracking the change in progressivity of a given tax scheme over time, requires a simple but robust methodology that produces a well-behaved measure of tax progressivity. We propose three key principles that describe a properly designed method for measuring tax progressivity and the proper behavior of its resulting tax-progressivity index.

1. The value of any tax-progressivity index should be independent of changes in the total level of income earned by the tax base, or changes in the total level of tax revenues collected from the tax base, as long as both of these underlying distributions remain unchanged across the entire tax base. As long as the tax-burden distribution remains unchanged, measuring relative tax progressivity should not be influenced by a change in a tax scheme's overall efficiency. If a higher percentage of total income is collected from the tax base (such as by eliminating deductions, exemptions, or other tax preferences), this should not influence the value of a tax-progressivity measure if the underlying tax-burden distribution remains unchanged. Likewise, a tax-progressivity measure should not be affected by economic growth alone if the underlying income distribution remains unchanged.

2. A tax-progressivity index value should include both income and tax-burden distributions simultaneously and *from across the entire population of the tax base.* Neither the changes in tax-burden distribution alone nor the changes in income distribution alone can necessarily reveal the degree to which a tax scheme's progressivity has changed. Either distribution can unilaterally affect the relative degree of tax progressivity so that both must be considered simultaneously. For example, if the upper-income groups grow proportionally richer but the underlying tax-burden distribution remains unchanged, the degree of tax progressivity has certainly declined.

Further, looking at only a segment of the income spectrum (such as top end of the tax base) to calculate tax progressivity yields only a partial—and therefore biased—picture of a tax scheme's overall degree of tax progressivity. This perspective is often justified on the basis that unequal income distribution gives a small subset of the population a large share of national income, and their influence on the degree of tax progressivity should therefore carry a greater weight when quantifying the overall degree of tax progressivity of a tax scheme. When determining tax-progressivity measures in this way, each income percentile's influence on the tax-progressivity calculation is weighted equally, regardless of the number of individuals in that income percentile. Such a rationale merely muddles the tax policy fairness debate.

To illustrate this point, consider two different ways to relate the ratio of tax shares paid to income shares earned across the nation's population when creating a tax-progressivity index. One approach conceptually lines up the entire *population* of an economy from highest to lowest income earner, allowing comparisons of the ratio of tax shares to income shares across different percentiles of the population. For example, if the top 10 percent of all income earners face a tax share that is 200 percent of their income share, while the bottom 10 percent of all income earners face a tax share that is only 25 percent of their income share, this information can reveal the magnitude of disproportionality with which the average tax rate rises with income across the population.

Another approach examines this same ratio of tax shares to income shares by conceptually lining up all the nation's *income* in the economy, as received by the lowest to highest income earners in the nation. This allows comparisons of the proportion of tax shares paid to income shares earned across the different percentiles of national income, rather than across the different percentiles of people. For example, assume that 10 percent of all national income received by the few highest income earners in the nation funds a tax share that is twice their 10 percent share of national income, while the 10 percent of all income received by the numerous lowest income earners in the nation funds a tax share that is only a quarter of their 10 percent share of national income. Using this approach, we *cannot* use this information to directly measure how disproportionately the average tax rate rises with people's income across the nation—at least not without adjusting somehow for the disproportionate decrease in population as we accumulate equal shares of the nation's income. Obviously the top 10 percent of national income was earned by people whose incomes were higher than those of the people who earned the next highest 10 percent of national income, but we don't know how much higher the incomes of those richest income earners were unless we divide each income percentile by the population in that percentile. But if we did that, then it would be far more efficient and direct to use the first approach discussed above.

Yet this second approach is the conceptual basis of anyone who draws conclusions about the degree of tax progressivity by examining only the impact that tax policy has had on the few top income earners in the nation, and justifies such conclusions because these few high-income earners represent a substantial amount of US national income. This second perspective assumes that as long as we account for the tax share impacts that involve a majority of the *income*, we can discount the tax share impacts that affect the majority of the *people* in the economy. As section V below demonstrates in more detail, one could come to very mistaken conclusions about tax progressivity using this perspective. How can we determine the rate at which tax progressivity increases with individual incomes across the population when we disconnect the income shares from the people who earn the income?

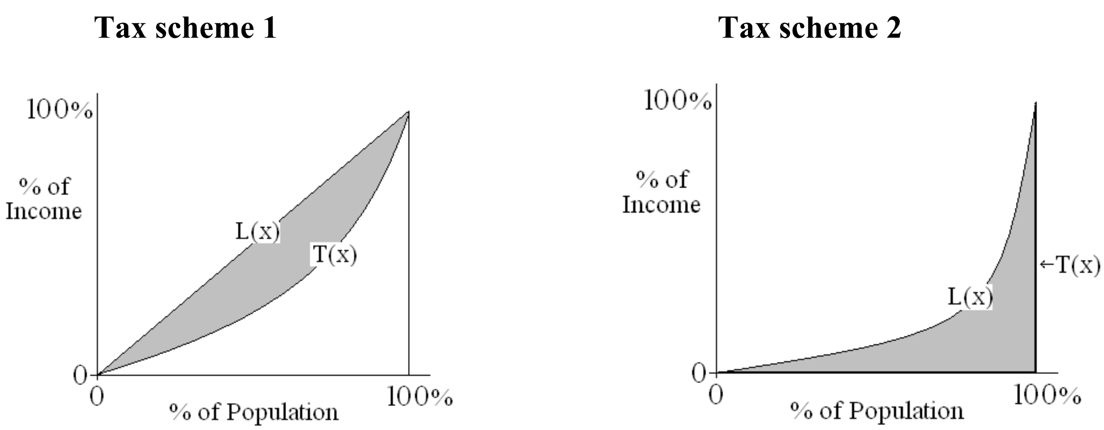
3. A tax-progressivity index should yield values that are well behaved across the entire spectrum of progressivity, so as to consistently yield a cardinal value estimate of a magnitude that accurately reflects the changing degree of tax progressivity. A proper tax-progressivity index should yield an intuitive and consistent interpretation when comparing two index values across different tax schemes, or when assessing changes in a given tax scheme over time.

Further, the index should produce cardinal values that are intuitively linked to the tax-progressivity concept being measured. This means that the distance between index values should retain a consistent meaning, as opposed to index values reflecting an ordinal ranking that only indicates whether progressivity increased or decreased.

IV. Designing an Index for Measuring Tax Progressivity

There is a simple but effective way to design a tax-progressivity metric that satisfies all three of these fundamental principles. The methodology is to combine information from the well-known Lorenz curve of income distribution with a similarly constructed tax-burden distribution curve, with both curves covering the entire tax base. To illustrate, a Lorenz curve is depicted by $L(x)$ and the tax-burden curve is depicted by $T(x)$ in two examples shown in figure 1.

Figure 1. Different Tax Schemes



In these graphs, the entire population of the tax base is organized from lowest to highest income along the *x* axis, as measured on a scale of 0–100 percent of the population. The entire nation's income appears on the y axis, as measured on a scale of 0–100 percent of all income earned by the population. Likewise, total federal tax revenues collected from the entire population can also be measured on the same *y* axis, on a scale of 0–100 percent. The Lorenz curve simply tracks the percentage of national income that is accumulated as we tabulate the entire population from lowest to highest income. Likewise, the tax-burden curve tracks the percentage of aggregate tax revenues that are accumulated as we tabulate the entire population by income.

Tax scheme 1 in figure 1 exhibits a near perfectly equal distribution of income across the entire population. When everyone earns almost exactly the same amount of income, each additional percentage of population adds the same additional percentage of total income. This scenario creates a linear Lorenz curve along the 45 degree line out of the origin. Tax scheme 2 exhibits a distribution of income in which the upper-income groups receive a disproportionate share of the total income earned in society, relative to the lower-income groups. This means that at lower levels of income, each additional percentage of the population adds *less* than a percent of additional income. However, among the higher-income groups, each additional percentage of population adds *more* than a percent of additional income. This results in the Lorenz curve being convex, bulging outward to the right. The more unequally income is distributed across society, the more convex the Lorenz curve.

Tax scheme 1 also exhibits a progressive tax rate scheme, where the tax-burden curve is located everywhere below the Lorenz curve. This occurs when the lower-income groups bear a smaller average tax rate than the upper-income groups. Starting at point (0, 0) and tabulating the population across the lower-income groups, each additional percentage of the population adds *less* to the total tax revenues collected than to the total income earned. In this range, the Lorenz curve climbs more steeply than the tax-burden curve. However, when tabulating the population across the upper-income groups, each additional percentage of the population adds *more* to the total tax revenues collected than to the total income earned. In this range, the tax-burden curve climbs more steeply than the Lorenz curve. Ultimately, both curves sum to 100 percent at the richest end of the population, where both curves terminate at point (1, 1).

Once the two curves are constructed, one can examine the interplay between them. Formby, Seaks, and Smith (1981) note that Suits and Kakwani independently introduced progressivity indexes involving the areas underneath the Lorenz and tax-burden curves. Stroup also introduced a progressivity index that uses the area between and under these curves. Which metric best describes tax progressivity and satisfies the three principles mentioned above? A concise description of each index follows.

Formby, Seaks, and Smith show that the Kakwani index is based on the difference in convexity between the Lorenz curve, $L(x)$, and the tax-burden curve, $T(x)$. Specifically, this metric can be expressed as twice the value of the shaded area in tax scheme 1 of figure 1. Equation 1a, below, is taken from Formby, Seaks, and Smith and shows how the Kakwani index is calculated:

(1a) $$\text{Kakwani index} = 2 \times [\text{area under } L(x) - \text{area under } T(x)].$$

Thus, its mathematical equation is

(1b) $$\text{Kakwani index} = 2 \times \left[\int_0^1 L(x)dx - \int_0^1 T(x)dx\right].$$

Formby, Seaks, and Smith also show that the Suits index can be expressed as a function of the shaded area between the same curves in tax scheme 1 in figure 1 above, but the difference in the convexity values of the Lorenz and tax-burden distributions is normalized by the slope value of the Lorenz curve at each income level, x. We start with the differential equation in Formby, Seaks, and Smith shown in equation 2a, below:

(2a) $$\text{Suits index} = 2\int_0^1 \left[x - T\left(L^{-1}(x)\right)\right]dx.$$

Note that here the integral is not from 0 percent to 100 percent of the population, but from 0 percent to 100 percent of the *income* earned across the tax base. However, we can monotonically transform the equation in terms of accumulating tax burden and income distribution across the population as follows:

(2b) $$\text{Suites index} = 2\int_0^1 [L(x) - T(x)]\, L'(x)dx.$$

The Stroup index measures tax progressivity as the *ratio* of the relative convexity values between the Lorenz and tax-burden distribution curves, rather than calculating their difference. This index normalizes the difference in the convexity values of the Lorenz and tax-burden distributions by expressing it as a ratio of the convexity of the Lorenz curve itself. The Stroup index can be expressed by equation 3a, below:

(3a) $$\text{Stroup index} = 1 - \frac{\text{Area under tax burden curve}}{\text{Area under Lorenz curve}}.$$

Thus, in calculus form, we have the following:

(3b) $$\text{Stroup index} = 1 - \frac{\int_0^1 T(x)dx}{\int_0^1 L(x)dx},$$

or, by rearranging, we get this equation:

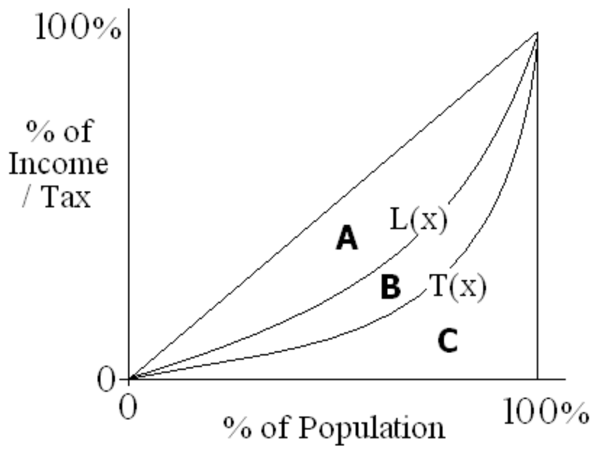
(3c) $$\text{Stroup index} = \frac{\int_0^1 L(x) - T(x)dx}{\int_0^1 L(x)dx}.$$

Now that each of these three index calculations can be shown to have an analog in the type of graph that appears in figure 1, the behavior of each index can be examined conceptually across the entire spectrum of tax progressivity. This behavior can be evaluated in light of the three principles proposed above to reveal each index's strengths and weaknesses in revealing the tax progressivity of a given tax scheme. After a brief discussion of how to interpret a tax-progressivity measure relating income and tax-burden distributions, we will examine each index.

V. Choosing the Best Index for Measuring Income Tax Progressivity

Consider the conceptual interpretation of an income-inequality measure known in the economics literature as the Gini coefficient. In a society where income is distributed with near perfect equality across the entire population, the Lorenz curve would be reflected by the 45-degree line connecting points (0, 0) and (1, 1), as illustrated by figure 2. As income distribution becomes more unequal, the Lorenz curve becomes more convex, separating itself ever farther from the 45 degree line. The Gini coefficient is an index relating how the Lorenz curve of actual income distribution diverges from this 45 degree line of near-perfect income equality.

Figure 2. A Typical Income and Tax-Burden Distribution Graph



Area A in figure 2 is the area under the line of near-perfect equal income distribution but above the Lorenz curve, $L(x)$. The Gini coefficient is simply the ratio of area A to the total area under the line of perfect income equality (the sum of areas A, B, and C). The more equally that income is distributed across society, the closer the Lorenz curve becomes to the 45 degree line, and the smaller area A becomes relative to the sum of areas A, B, and C. This means that as income is equalized, area A disappears and the value of the Gini coefficient approaches zero.

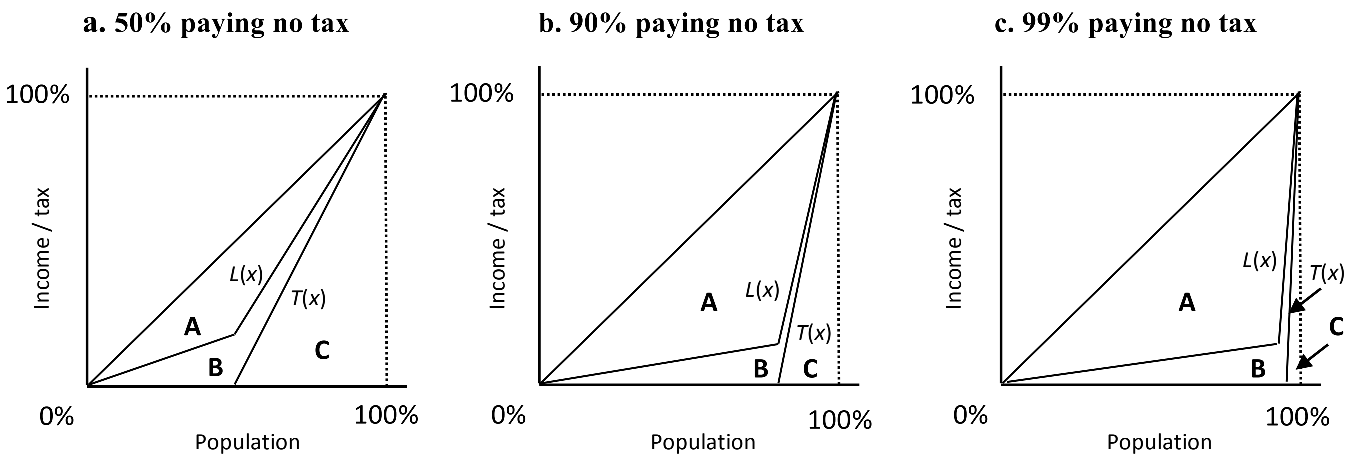
Further, as income becomes more unequally distributed in society, the Lorenz curve becomes more convex and area A becomes ever larger relative to the sum of areas A, B, and C. This means that as society approaches the extreme income inequality of a single individual earning nearly all the income in society, areas B and C disappear and area A converges to the entire area under the 45 degree line. At this point the Gini coefficient obtains a maximum value of 1.0. Thus, the spectrum of possible Gini coefficient values is easily interpreted as a cardinal scale of index values reflecting the degree of income inequality as it increases from zero (perfect equality) to one (perfect inequality). This conceptual framework helps us test the consistency of each tax-progressivity methodology.

The Kakwani index. This index sums the vertical difference between $L(x)$ and $T(x)$ across the entire x axis, and the value of this index varies with the size of area B in figure 2. In the case of a proportional income tax (or "flat" tax), everyone pays the same proportion of income in tax. This means the tax-burden curve, $T(x)$, coincides perfectly with the Lorenz curve, $L(x)$. Every additional percentage of population adds the same percentage to income as to taxes. This means area B disappears and the Kakwani index approaches a value of zero. However, this index does not behave well as tax progressivity increases toward its maximum value.

Consider again the two tax schemes in figure 1. It is possible for the shaded area in tax scheme 1 to be exactly equal in value to the shaded area in tax scheme 2. This implies that the Kakwani index would produce the exact same value of progressivity in both tax schemes. Yet tax scheme 1 exhibits a nearly linear Lorenz curve combined with an only moderately convex tax-burden curve. Here, all individuals bear a share of the tax burden, though their shares rise disproportionately with income. Tax scheme 2 exhibits a Lorenz curve that is more convex, but also a tax-burden curve that is a horizontal line until it becomes nearly vertical at the single-richest person in the tax base. Here, far more people face a lower average tax rate because only one person bears the entire tax burden. This means tax scheme 2 is much more progressive, yet the Kakwani index produces the same index value for both tax schemes. The inconsistent behavior of this index across the tax-progressivity spectrum violates principle 3.

The Suits index. Returning to figure 2, the Suits index may also be viewed as summing the vertical difference between $L(x)$ and $T(x)$ for every value of x in area B, but with each difference value being multiplied by the slope of $L(x)$ at that value of x. Whereas the Kakwani index sums the difference in convexity between these two curves equally across the entire tax base, the Suits index weights the difference for the upper-income end of the spectrum more heavily than in the lower-income end. This is a potential source of bias, as illustrated by the series of tax schemes represented in figure 3, below. As the tax schemes change from figure 3a to figure 3c, they portray an increasingly smaller portion of the population bearing an ever larger share of the total tax burden, but also enjoying an ever larger share of total income.

Figure 3. A Comparison of Various Income and Tax-Burden Distributions



For example, assume the economy comprises 100 people. In figure 3a, the poorest 50 people pay no income tax and earn 10 percent of all income. The remaining 50 people split the entire tax burden evenly and split the remaining 90 percent of all income evenly. This means each of these 50 people faces the same average tax rate, with each person bearing a ratio of tax-burden share to income share of 1 to 0.9. In figure 3c, 99 of the 100 people pay no taxes at all and equally split 10 percent of all income. The remaining lone taxpayer earns 90 percent of all income and bears the entire tax burden.

Although the entire taxpaying population in both figure 3a and figure 3c all pay the same 1 to 0.9 ratio of tax burden to income share, the average tax rates are lower for 49 of the people (individuals 51 to 99) in figure 3a. Therefore, tax progressivity necessarily increases as the tax scheme goes from figure 3a to figure 3c. Because the normalizing weights in the Suits index increase proportionately with income across the individuals of the population, the Suits index perceives the tax schemes in figure 3a through figure 3c as having the exact same level of overall progressivity, which violates principle 3.

The Piketty and Saez methodology. The tax-scheme examples of figure 3 also illustrate the potential bias that lies within the methodology used by Piketty and Saez to assess overall federal tax progressivity. When comparing data from 1960 to 2004, they focus on the declining average tax rate facing the highest 1 percent of income earners to claim that federal tax policy in general has declined during this period. However, their data also indicate that a lower-income segment with a much larger population also experienced falling average tax rates over this period. They allow the influence of those few with the greatest incomes in one population segment to prevail in determining the degree of overall tax progressivity simply because they command a much larger portion of national income than the more populous but poorer income segment.

Piketty and Saez effectively give a larger weight to high-income earners, just as in the Suits index, but fail to disclose their specific weighting scheme. Recall that between the tax schemes in figure 3a and figure 3c, the same proportion of income dollars (90 percent) paid the same amount of tax burden (100 percent). Yet any tax scheme like that in figure 3c, where 88 percent more people pay a lower average tax rate and *nobody* pays a higher average tax rate, must be labeled as having a lower overall degree of tax progressivity. If the influence of a segment of the population that has earned a predetermined percentage of national income should be chosen to dominate the calculation of overall tax progressivity for a given tax base, the number of people in that specific percentage is not specified, and this method cannot yield a cardinal measure of progressivity. This method violates principles 2 and 3.

The Stroup index. Again referring to figure 2, the Stroup index value is created by taking the ratio of the area between the Lorenz and tax-burden curves (area B) to the total area under the Lorenz curve (the sum of areas B and C). Note that this construct mirrors the conceptual structure of the Gini coefficient, as discussed earlier. In the case of a proportional income-tax (or flat-tax) scheme, the two curves converge as the entire population shares the tax burden proportionally to their income. In this case, the value of the Stroup index approaches zero as area B disappears. In the case of maximum tax progressivity where one individual bears the entire tax burden, the area between the Lorenz and tax-burden curves approaches equality with the total area under the Lorenz curve. As area C disappears, the value of the Stroup index approaches one.

Further, the Stroup index correctly identifies tax scheme 2 in figure 1 as having a higher level of progressivity. It also identifies the tax scheme in figure 3c as having the highest level of tax progressivity among the three scenarios. In fact, the Stroup index smoothly and monotonically increases from its lowest possible value of tax progressivity (0.0) to its highest possible value (1.0) in a manner that satisfies the conceptual expectations of how tax progressivity changes across its spectrum. Table 1, below, illustrates the value of the Kakwani, Suits, and Stroup indexes for the three different tax schemes depicted in figure 3a through figure 3c. This table reveals how the value of the Stroup index monotonically increases with the percentage of the tax base that is bearing no tax burden, while the other two indexes do not.

Table 1. Comparison of Various Income and Tax Burden Distributions

Figure	Percentage of pop. earning 10% of all income while paying *no* tax	Percentage of pop. earning 90% of all income while paying entire tax burden	Value of Kakwani index	Value of Suits index	Value of Stroup index
3a	50%	50%	0.10	0.10	0.17
3b	90%	10%	0.10	0.10	0.50
3c	99%	1%	0.10	0.10	0.91

The above empirical analysis reveals that as a given tax scheme becomes more progressive and the tax-burden curve becomes ever more convex relative to the income-distribution curve, the reliability of the Kakwani and Suits indexes to accurately reflect progressivity becomes increasingly suspect because their mathematical constructs fail to properly reflect traditional tax-progressivity concepts, as discussed above. Next, we examine the commonly accepted methodology for estimating the value of these tax-progressivity indexes from empirical data.

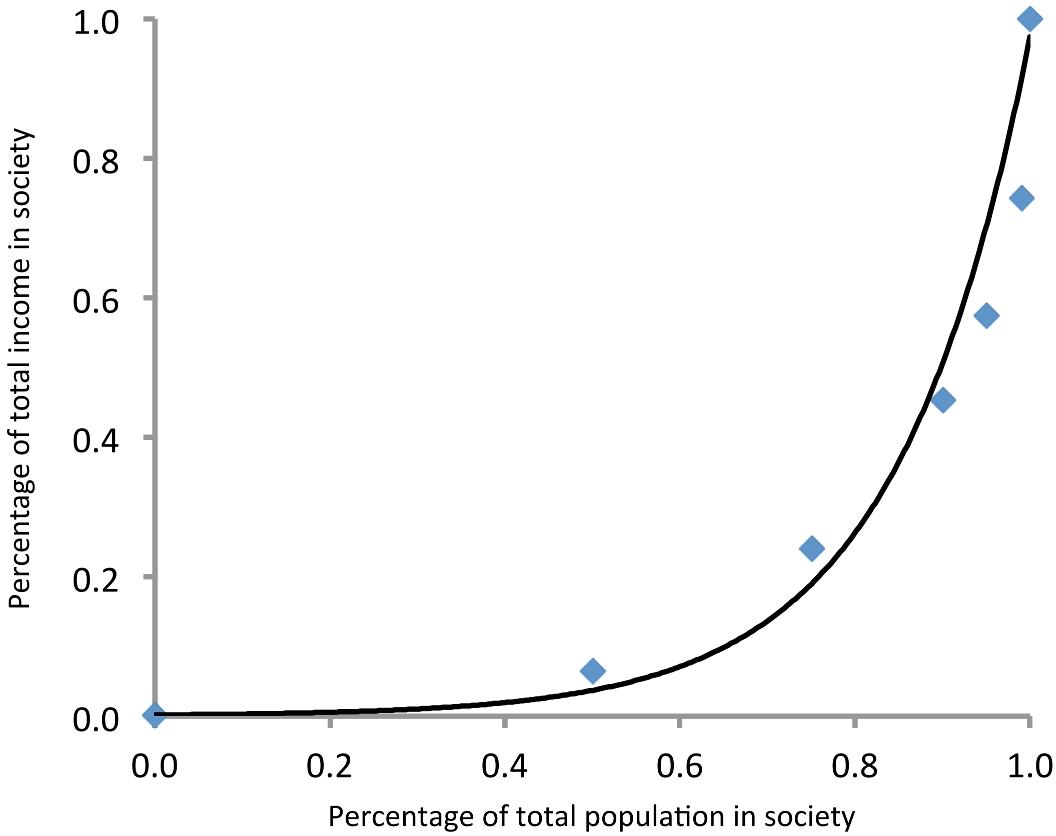
VI. Choosing the Best Estimation Method

We will use the same IRS dataset on AGI and income tax used to estimate the values of the different tax-progressivity indexes in table 2 to illustrate how well different estimation processes perform in generating the underlying Lorenz and tax-burden curves for these index values. We estimate all three progressivity indexes using the annualized data of AGI and federal income tax revenues collected for the entire US federal income tax base.

This dataset provides the annual distribution data necessary to generate cumulative data points for both AGI and income tax curves at the 50 percent, 75 percent, 90 percent, 95 percent, and 99 percent levels of the population (see appendix A). The bottom half of all income earners pays less than 3 percent of all federal income taxes collected, which may explain why data points for the 10 percent or 25 percent population levels are not provided. When combined with the 0 percent and 100 percent endpoints of both curves, this dataset generates a total of seven observation points with which to estimate an equation for each of the two distribution curves.

These income and tax-burden data do not fit well with standard mathematical models for estimating nonlinear curves. A simple quadratic, exponential, or polynomial equation does not fit either curve very well when it must include the endpoints (0, 0) and (1, 1). To illustrate, we use the seven observation points for the 1986 IRS federal income tax data in a simple exponential model to estimate the tax-burden curve, $T(x)$, in figure 4, below. This figure reveals that this method tends to underestimate the true underlying curve at the lower-level data points (the 0 percent, 50 percent, and 75 percent levels) and overestimate it the upper-level data points (the 90 percent, 95 percent, and 99 percent levels). Further, if the estimation process for this functional form starts at point (0, 0), it necessarily misses the endpoint (1, 1). If the estimation process starts at point (1, 1), it necessarily misses the starting point (0, 0).

Figure 4. Estimating the Lorenz Curve



An alternative method for estimating the Lorenz and tax-burden curves would be to fit a linear spline function to connect each data point to the next via a straight line. This method hits both endpoints of the curve, but it overestimates the area under the curve between each pair of data points. It also creates curves that do not increase smoothly with the accumulation of population across the tax base along the *x* axis. Therefore, we propose that the best methodology for estimating the curves used for calculating tax-progressivity indexes should exhibit the following fundamental properties:

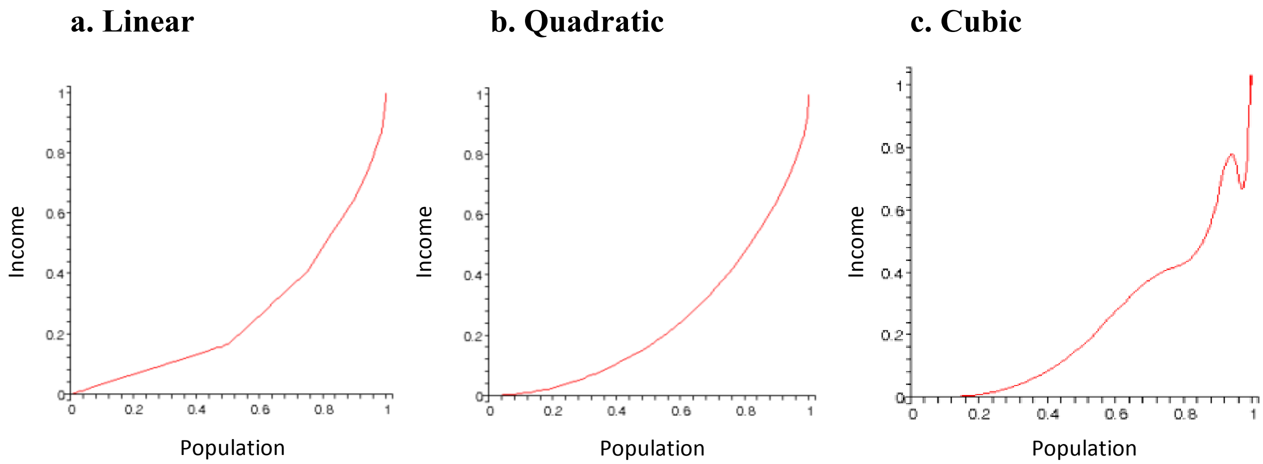
1. Avoid any known bias when estimating either the Lorenz curve or the tax-burden curve. Such biases can be avoided, in part, if the estimated curve passes through each and every data point, including both the origin point (0, 0) and the culmination point (1, 1). Further, the estimation process should not be consistently biased in estimating the slope between these data points.

2. Allow the slope of the Lorenz and the tax-burden curve estimates to increase continuously across the entire tax base, avoiding any sharp corners at known data points. This creates a well-behaved function describing curves that will, in turn, create a well-behaved change in the value of the index as tax progressivity changes. It also incorporates all available information into the curve estimates, which will be evident in the tax-progressivity index itself.

3. The slope of both curves should have a value of zero at the origin points (0, 0). The smallest, bottom fraction of the population has no measurable income or tax burden, so an arbitrarily small increase from 0 percent should not raise the income or tax-burden values at all.

The polynomial spline interpolation method satisfies all three of these criteria. A technical discussion explaining this process appears in appendix B. This process can utilize a linear equation to fit each observation point in the data, or it can use polynomial equations such as quadratic or cubic formulas. To illustrate, the same Lorenz curve is estimated with the aforementioned IRS data from 1986 in all three graphs in figure 5, below. Each graph reflects the Lorenz curve estimated with linear, quadratic, and cubic interpolation methods, respectively.

Figure 5. Examples of Different Interpolation Approaches



Cubic spline interpolation. This estimation technique was first advocated for modeling Lorenz curves by Paglin (1975), and many others later used it for estimating tax-burden curves to calculate progressivity measures. For example, Formby, Seaks, and Smith (1981) referenced Paglin's work and claimed the "cubic spline technique is more accurate than the conventional straight line method used by Suits." However, using a cubic spline interpolation contains serious modeling errors not prevalent when using a quadratic interpolation.

Different mathematical equations can be used in the interpolation process to estimate the Lorenz curve. The coefficients in each polynomial equation are what determine the basic shape of each curve. Figure 5, above, reveals how three different polynomial equations are used with the same interpolation process to connect the data points in order to create three estimates of the same Lorenz curve.

Figure 5c in particular reveals the erratic shape of the estimated curve that results when a cubic polynomial formula is used with the interpolation process to estimate the Lorenz curve. One way to interpret this result is to recognize that in order to keep the interpolation function smoothly continuous from one data point to the next, the coefficients of the cubic polynomial functions tend to get larger and larger, as they try to fit one data point to the next. Just as a driver who overcorrects his turns navigating an icy road can easily make a bad situation even worse, the cubic polynomial equation progressively loses any semblance of modeling a well-behaved Lorenz curve as it tries to fit the curve to each successive data point. This example reveals how the widely accepted cubic interpolation methodology does not always create a properly convex Lorenz curve (one bowed outward to the right from the origin) or a tax-burden curve that monotonically increases (rises consistently) over the entire tax base. This estimation behavior violates principle 2.

Linear spline interpolation. The Lorenz curve estimated by linear interpolation in figure 5a is better behaved, increasing monotonically across the entire tax base. This is a popular method and Suits has used it to estimate his tax-progressivity index, while Piketty and Saez have used it to estimate their increasing average tax rate function across all income levels of the population. However, the Lorenz curve it creates suffers from a known bias: it overestimates the areas between each data point, relative to the real Lorenz curve. This estimation behavior violates principle 1.

Quadratic spline interpolation. The Lorenz curve derived from the quadratic interpolation method in figure 5b is both monotonically increasing and consistently convex between data points over the entire tax base. Compared to the popularly used methodologies for estimating tax-progressivity indexes, the quadratic spline interpolation methodology satisfies all three fundamental principles as the superior method for estimating the value of a tax-progressivity index. Next, we use this method to fit the annual IRS data of AGI and income tax revenue shares to illustrate the relative behavior of the three tax-progressivity indexes.

VII. Letting the Data Speak

We illustrate the quadratic spline interpolation method with the 1986 AGI and tax-burden data to calculate the Stroup index using equation 3 from above. This equation reveals that the area under the Lorenz curve (area B in figure 2) is 0.2519, and the area under the tax-burden curve (the sum of areas B and C in figure 2) is 0.1549. The estimated value of the Stroup index for 1986 is 0.3849.

(3) $$S = \frac{0.2519 - 0.1549}{0.2519} = 0.3849.$$

We apply the same quadratic spline interpolation method to the IRS data from 1986 to 2009 to construct the values of all three tax-progressivity metrics annually. These results appear in table 2, below. The AGI column reveals the values for the area under the Lorenz curve, and the tax column refers to the values of the area under the tax-burden curve. The three remaining columns display the values for the Kakwani, Suits, and Stroup indexes, respectively. The plus and minus signs indicate whether the index on the left increased (became more progressive) or decreased (became less progressive) from the year before.

Table 2. Comparison of the Estimated Values of the Different Indexes

Year	AGI	Tax	Kakwani	Up or down	Suits	Up or down	Stroup	Up or down
1986	0.2519	0.1549	0.1939		0.2776		0.3849	
1987	0.2425	0.1509	0.1833	−	0.2617	−	0.3778	−
1988	0.2326	0.1447	0.1758	−	0.2483	−	0.3779	+
1989	0.2340	0.1489	0.1702	−	0.2378	−	0.3636	−
1990	0.2349	0.1497	0.1702		0.2376	−	0.3624	−
1991	0.2368	0.1479	0.1778	+	0.2503	+	0.3753	+
1992	0.2332	0.1405	0.1853	+	0.2633	+	0.3974	+
1993	0.2336	0.1360	0.1951	+	0.2838	+	0.4177	+
1994	0.2329	0.1351	0.1957	+	0.2844	+	0.4201	+
1995	0.2288	0.1308	0.1960	+	0.2876	+	0.4282	+
1996	0.2231	0.1250	0.1962	+	0.2897	+	0.4396	+
1997	0.2188	0.1230	0.1917	−	0.2806	−	0.4379	−
1998	0.2155	0.1179	0.1954	+	0.2875	+	0.4532	+
1999	0.2107	0.1132	0.1951	−	0.2896	+	0.4629	+
2000	0.2066	0.1104	0.1924	−	0.2847	−	0.4656	+
2001	0.2180	0.1172	0.2015	+	0.2959	+	0.4622	−
2002	0.2233	0.1127	0.2211	+	0.3258	+	0.4953	+
2003	0.2205	0.1121	0.2168	−	0.3200	−	0.4916	−
2004	0.2128	0.1057	0.2141	−	0.3176	−	0.5031	+
2005	0.2047	0.0992	0.2110	−	0.3140	−	0.5155	+
2006	0.2010	0.0976	0.2067	−	0.3074	−	0.5144	−
2007	0.1978	0.0960	0.2038	−	0.3024	−	0.5150	+
2008	0.2060	0.0983	0.2155	+	0.3214	+	0.5230	+
2009	0.2158	0.0946	0.2425	+	0.3660	+	0.5616	+

It is revealing to compare the disparate trends that these three index values track over the years when using the same dataset. The generally falling values in the AGI column indicate that the area under the Lorenz curve has diminished over time. This finding supports claims that US income distribution has become ever more unequal during this time. It also supports the concerns cited by those who believe that federal income tax progressivity may have decreased over this period. Yet the falling values in the tax column indicate that the area under the tax-burden curve has also diminished over time. This finding supports claims that the federal income tax burden distribution has become increasingly unequal across US taxpayers, and it supports the concerns cited by those who believe that federal income tax progressivity has increased over this time period.

However, income and tax-burden distributions need to be assessed simultaneously to determine the change in tax progressivity of federal income tax policy over time. If the magnitude of increased income inequality surpasses the magnitude of increased tax-burden inequality, this indicates that tax progressivity has *decreased* overall, despite the growing tax-burden gap between the rich and poor. On the other hand, if the magnitude of increased tax-burden inequality surpasses the magnitude of the increased income inequality, this indicates that tax progressivity has *increased* overall, despite the widening income gap between the rich and poor. If a federal income tax policy analyst trusted the underlying data used to create these numbers, he or she still must turn to the values of one of these three indexes to determine the degree to which federal income tax progressivity has changed.

Though all three indexes produce different index values each year, these values (out to three decimal places) have generally fallen from 1986 to 1990 and have generally risen from 1990 to 2002. However, looking at the period under the Jobs and Growth Tax Relief Reconciliation Act from 2003 to 2009, the Kakwani and Suits indexes both fell slightly each year before finally rising again in 2008. The Stroup index, meanwhile, generally *increased* from 2003 until 2008, before jumping up substantially in 2009.

If one accepts the dataset as faithfully representing federal income tax progressivity, which index can be most trusted to accurately reflect the true change in federal income tax progressivity during this time? It depends on how well each index can be trusted to yield values that are well-behaved across the entire spectrum of progressivity so as to consistently yield a cardinal value estimate of magnitude that accurately reflects the changing degree of tax progressivity across the entire tax base. Based on the above analysis, only the Stroup index can make that claim.

VIII. Conclusion

The federal tax-burden fairness debate rages on, partially fed by a lack of positive claims about overall federal tax-burden progressivity that can be tested—and thereby trusted—to reveal the true degree of tax progressivity of a given tax scheme. What this debate needs is a reliable and accurate measure of tax progressivity that can be trusted to reveal the true difference in progressivity across competing tax schemes and that faithfully tracks the changes in progressivity of a tax scheme over time.

While the broadly accepted tax-progressivity analysis of Piketty and Saez uses a reasonable dataset of income and tax-burden distribution across the American tax base, the authors' attempt to gain insights into the issue of federal tax-burden fairness has shortcomings. The above discussion reveals that their methodology for assessing the degree of federal tax-burden progressivity is potentially misleading. Their claim that federal tax policy has become decidedly less progressive is justified on an observed decline in the average federal tax rate among the top 1 percent of taxpayers, but they ignore the concomitant decline in average tax rate enjoyed by the taxpayers in the 20th to 40th percentiles. Their lack of an explicit method for comparing the magnitudes of income and tax-burden distribution changes across the entire tax base cannot quantify the degree of change in tax progressivity. This makes their claim about federal tax policy progressivity a subjective statement in itself that is inherently untestable. What is needed is a positive, testable claim that can inform the normative debate over federal tax policy fairness.

Further, the well-defined and broadly accepted measures of tax progressivity by Kakwani and Suits fail to stand up to reasonable expectations for how a tax-progressivity index should behave across the possible spectrum of tax progressivity. Only the Stroup index behaves appropriately, meaning that it could properly utilize a well-defined income and tax-burden distribution dataset to properly estimate the effective magnitude of tax-progressivity changes. For example, the average tax rate data derived by Piketty and Saez could be used with the Stroup index to create a federal tax-progressivity index that yields cardinal values reflecting the degree of federal tax progressivity that could be observed over time or across tax schemes.

A robust methodology for assessing tax progressivity leads to a meaningful tax-progressivity index, which in turn allows for a more edifying debate over tax policy fairness. This process can generate positive claims about changes in federal tax policy progressivity, where the claims are testable and rhetorically defensible. Such positive statements about tax-scheme progressivity would enlighten the normative public debate over federal tax policy fairness and help break through its current rhetorical impasse. The conceptual framework and analytical clarity discussed above are what is needed to support an index of tax-scheme progressivity that can enlighten the federal tax-burden fairness debate and help overcome the prevailing rhetorical gridlock that prevents a consensus view for designing an optimal federal tax policy.

Appendix A. Internal Revenue Service Data on Cumulative Federal Income Tax Collected and Adjusted Gross Income Earned across Households, by Year

Percentile of Tax Base Population

AGI	50th	75th	90th	95th	99th
1986	0.167	0.410	0.649	0.759	0.887
1987	0.156	0.393	0.631	0.743	0.877
1988	0.149	0.376	0.606	0.715	0.848
1989	0.150	0.377	0.610	0.722	0.858
1990	0.150	0.379	0.612	0.724	0.860
1991	0.151	0.382	0.618	0.732	0.870
1992	0.149	0.375	0.608	0.720	0.858
1993	0.149	0.376	0.610	0.722	0.862
1994	0.149	0.374	0.608	0.722	0.862
1995	0.145	0.366	0.598	0.712	0.854
1996	0.141	0.357	0.584	0.696	0.840
1997	0.138	0.350	0.572	0.682	0.826
1998	0.137	0.344	0.562	0.672	0.815
1999	0.133	0.335	0.551	0.660	0.805
2000	0.130	0.329	0.540	0.647	0.792
2001	0.138	0.348	0.569	0.680	0.825
2002	0.142	0.356	0.582	0.695	0.839
2003	0.140	0.351	0.576	0.688	0.832
2004	0.134	0.339	0.557	0.666	0.810
2005	0.128	0.325	0.536	0.643	0.788
2006	0.125	0.318	0.527	0.633	0.779
2007	0.123	0.313	0.520	0.626	0.772
2008	0.128	0.326	0.542	0.653	0.800

Percentile of Tax Base Population

Tax	50th	75th	90th	95th	99th
1986	0.065	0.240	0.453	0.574	0.743
1987	0.061	0.231	0.444	0.567	0.752
1988	0.057	0.222	0.427	0.544	0.724
1989	0.058	0.228	0.442	0.561	0.748
1990	0.058	0.230	0.446	0.564	0.749
1991	0.055	0.227	0.442	0.566	0.752
1992	0.051	0.215	0.420	0.541	0.725
1993	0.048	0.207	0.408	0.526	0.710
1994	0.048	0.205	0.406	0.525	0.711
1995	0.046	0.196	0.393	0.511	0.697
1996	0.043	0.187	0.375	0.490	0.677
1997	0.043	0.183	0.368	0.481	0.668
1998	0.042	0.173	0.350	0.462	0.653
1999	0.040	0.165	0.336	0.446	0.638
2000	0.039	0.160	0.327	0.435	0.626
2001	0.040	0.171	0.351	0.468	0.661
2002	0.035	0.161	0.343	0.462	0.663
2003	0.035	0.161	0.342	0.456	0.657
2004	0.033	0.151	0.318	0.429	0.631
2005	0.031	0.140	0.297	0.403	0.606
2006	0.030	0.137	0.292	0.399	0.601
2007	0.029	0.134	0.288	0.394	0.596
2008	0.027	0.137	0.301	0.413	0.620

Note: AGI stands for adjusted gross income; Tax stands for cumulative federal income tax collected.

Source: David S. Logan. “Summary of Latest Federal Individual Income Tax Data.” Tax Foundation. October 24, 2011. www.taxfoundation.org/taxdata/show/250.html.

Appendix B. Illustrating the Polynomial Interpolation Process for Estimating Index Values

Rather than a single polynomial being fitted to all the points of a given curve, a degree n polynomial equation can be fitted between each pair of known data points along the curve being estimated, keeping $n - 1$ derivatives continuous at each point. This process is repeated sequentially across all the data points until a piecewise equation is formed that describes the entire curve. The elegance of this methodology is that it generates a unique model.

For example, consider a cubic polynomial used to approximate an actual income curve between two data points. The curve must go through the two points and both the first and second derivatives are required to be the same as the curve to the left of the curve in question. This takes up the four degrees of freedom in a cubic polynomial, leaving a unique curve with two desirable characteristics. The curve is clearly twice differentiable and exactly matches the data at all measurement points. (Differentiability is a desirable characteristic since a large underlying population that is ordered by increasing income would yield a differentiable curve if all data were considered in the calculation.)

We will now calculate the Lorenz curve for 1986 using the raw data points (0, 0), (0.5, 0.1666), (0.75, 0.4096), (0.9, 0.6488), (0.95, 0.7589), (0.99, 0.8870), and (1, 1). This means that 0 percent of the US population earned 0 percent of aggregate gross income in 1986, that 50 percent of the US population earned 16.66 percent of aggregate gross income in 1986, and so on. A polynomial interpolation calculates six separate polynomials between the seven data points listed above. The polynomial functions would be P_1 on the interval from 0 to 0.5, P_2 on the interval from 0.5 to 0.75, P_3 on the interval from 0.75 to 0.9, P_4 on the interval from 0.9 to 0.95, P_5 on the interval from 0.95 to 0.99, and P_6 on the interval from 0.99 to 1.

Linear polynomials are easiest to fit, since two endpoints uniquely determine a line segment. Below are the coefficients for the linear interpolation of the 1986 Lorenz curve:

$$
\begin{aligned}
P_1 &= 0.33\,x + 0,\\
P_2 &= 0.97\,x + -0.32,\\
P_3 &= 1.59\,x + -0.79,\\
P_4 &= 2.20\,x + -1.33,\\
P_5 &= 3.20\,x + -2.28,\\
P_6 &= 11.30\,x + -10.30.
\end{aligned}
$$

Note that $P_1(0.5) = P_2(0.5)$, and so on.

We now consider the seven data points that inform the 1986 Lorenz curve. They may be interpolated into a quadratic spline. Begin by choosing a quadratic polynomial that passes through the first two data points: 0 percent at the 0 percentile (0, 0), and 16.66 percent at the 50th percentile (0.5, 0.1666). Next, note that the polynomial must have slope 0 at point (0, 0), which adheres to the methodological concerns outlined above. Therefore, a quadratic equation, $f(x) = ax^2 + bx + c$, is required to describe the function that satisfies the following:

(1) $f(0) = 0.$ The value of the function at the 0 percentile is 0 percent.

(2) $f(0.5) = 0.1666.$ The value of the function at the 50th percentile is 16.66 percent.

(3) $f'(0) = 0.$ The value of the first derivative at the 0 percentile is 0.

These three requirements amount to the following three algebraic equations:

(4) $$0 = a \cdot 0^2 + b \cdot 0 + c,$$

(5) $$0.1666 = a \cdot 0.5^2 + b \cdot 0.5 + c,$$

(6) $$0 = 2a \cdot 0 + b.$$

This system of three independent equations and three unknown variables has a unique solution for *a*, *b*, and *c*. Therefore, a unique quadratic equation fits all three of these requirements. In the example above, $a = 0.67$, $b = 0.00$, and $c = 0.00$. This polynomial dictates what the rate of increase (or slope, or derivative) of our estimated curve must be when the value of $x = 0.5$. Specifically,

(7a) $$f'(x) = 2ax + b = 1.34x + 0,$$

(7b) $$f'(0.5) = 0.67.$$

To build the next piece of the curve, couple the derivative information above with the next pair of points: (0.5, 0.1666) and (0.75, 0.4096). This creates another system of three independent equations:

(8) $$0.1666 = a \cdot 0.5^2 + b \cdot 0.5 + c,$$

(9) $$0.4096 = a \cdot 0.75^2 + b \cdot 0.75 + c,$$

(10) $$0.67 = 2a \cdot 0.5 + b.$$

The coefficients that solve this system of equations yield the polynomial that fits the data perfectly between $x = 0.5$ and $x = 0.75$. Continuing in a similar manner, a unique polynomial can be found between each successive pair of data points, with a slope that matches the slope of the previous polynomial at the adjoining endpoint. The end result is a unique mathematical model that matches every data point, has a continuously increasing value, and has a zero slope at $x = 0$. It also makes calculating the area under the curve rather simple, reducing the area calculation to polynomial integration.

Below are the results for the quadratic interpolation for the 1986 AGI data.

$$
\begin{aligned}
P_1 &= 0.67\,x^2 + 0\,x + 0,\\
P_2 &= 1.22\,x^2 + -0.56\,x + 0.14,\\
P_3 &= 2.11\,x^2 + -1.89\,x + 0.64,\\
P_4 &= 5.81\,x^2 + -8.54\,x + 3.63,\\
P_5 &= 17.76\,x^2 + -31.24\,x + 14.42,\\
P_6 &= 738.73\,x^2 + -1{,}458.77\,x + 721.04.
\end{aligned}
$$

If it is stipulated that the first and second derivatives of the polynomial have a value of zero when $x = 0$, then a cubic spline can be fit in a similar manner. Using the dataset above, the cubic interpolation would be as follows:

$$
\begin{aligned}
P_1 &= 1.33\,x^3 + 0\,x^2 + 0\,x + 0,\\
P_2 &= -8.44\,x^3 + 14.66\,x^2 + -7.33\,x + 1.22,\\
P_3 &= 81.20\,x^3 + -187.04\,x^2 + 143.94\,x + -36.60,\\
P_4 &= -1{,}603.23\,x^3 + 4{,}360.92\,x^2 + -3949.22\,x + 1{,}191.35,\\
P_5 &= 9{,}835.54\,x^3 + -28{,}239.56\,x^2 + 27{,}021.24\,x + -8{,}615.96,\\
P_6 &= -247{,}652.95\,x^3 + 736{,}501.25\,x^2 + -730{,}072.16\,x + 241{,}224.86.
\end{aligned}
$$

Note the problematic negative lead coefficients and note also how the coefficients grow exponentially larger. Figure 5 (page 28) demonstrates these problems graphically.

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